Clumping instability of a falling horizontal lattice

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Particle falling through a viscous liquid tend to form clumps under the influence of viscous forces. An analysis of this effect for a falling horizontal lattice indicates that the clumps which form are much larger than expected on the basis of nearest neighbor approximations. This difference in behavior can be traced to the increasing influence of distant neighbors in two dimensions, and suggests that the clumping instability in three dimensions may not be described as a simple extension of the one-dimensional instability.

I. INTRODUCTION

Two spheres moving through a liquid are often subject to forces which tend to move the particles relative to each other. The best known of these forces is the Bernoulli pressure, which brings the two particles together if they fall side by side, or forces the trailing particle into the wake of the leading particle if one of them takes the lead. The Bernoulli effect is most noticeable at high velocities, being proportional to V^2. These forces therefore tend to vanish when the particles are moving slowly (or at low Reynolds number), and the particles do not move relative to each other.

At low Reynolds numbers, the principal force acting on the particles is the Stokes drag force given by

$$F_d = -6\pi\mu a U, \tag{1}$$

where μ is the viscosity of the liquid. If two particles are falling together, and separated by a distance $R \gg a$, the Stokes drag on each of them is reduced by the presence of the other, and the two particles will fall faster. This reduces the effective drag force to[1]

$$F_d = -6\pi\mu a U(1 - 3a/4R). \tag{2}$$

This reduction in the drag force occurs whether the particles are falling side by side or at an angle to the horizontal.

An additional force appears in the low Reynolds limit which is influenced by the orientation of the two particles. This force, known as the line-of-centers force, acts equally on both particles, and has the magnitude[1]

$$F_{lc} = \tfrac{9}{2}(\pi\mu a^2 U/R)\cos\theta. \tag{3}$$

This force acts in the downward direction along the line joining the centers of the two particles.

Neither the modified Stokes drag nor the line-of-centers force tends to alter the relative positions of the two particles, because both forces act equally on each particle. Consequently, two particles drifting through a viscous liquid in the low Reynolds number regime will not tend to clump or to spread as a result of fluid forces.

When more than two particles are present, a new possibility for instability of the orientation arises. Consider three particles drifting through a viscous liquid as shown in Fig. 1. At low Reynolds number, Bernoulli forces are negligible, and only the Stokes drag and line-of-centers force need to be considered.

The drag force is reduced by the presence of neighbors, and since the center particle has two neighbors, while the outside particles have only one each, the center particle experiences less drag, and tends to fall faster, thus moving even farther ahead of its neighbors. The line-of-centers force acting on the two outer particles is due mainly to the influence of the center particle, and tends to move the outer particles over toward the center of the cluster. The line-of-centers force on the central particle is the vector sum of two contributions, and tends to increase its rate of fall even more. Thus, the original cluster of three particles will change its shape, with the central particle moving farther ahead, while the outer particles approach each other behind the central one. If a horizontal line of particles is falling through a viscous liquid, triplet clusters of this type tend to develop and disrupt the initially uniform spacing of the particles, as pointed out in a previous paper.[2] The clusters that develop from a one-dimensional array of particles will most likely be spaced about four times the original particle spacing, and will develop when the array has drifted a distance which is perhaps ten times the original particle spacing.

This result was originally predicted and confirmed only for a line of particles, but it has since been used to explain clumping instabilities in three-dimensional processes such as bioconvection,[3] sedimentation,[4] pneumatic transport,[5] and suspension flow.[6] While this may appear to be a natural extension, it should be pointed out that not all uniform arrangements exhibit the instability. An infinite vertical line of particles, for example, is stable against clumping (although the ends of a vertical line may not be).[7] Since three-dimensional

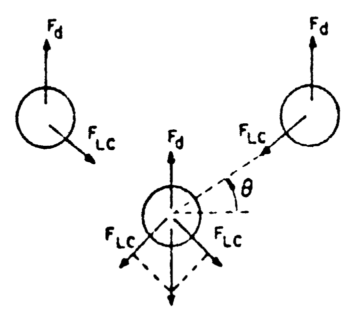

FIG. 1. Three spheres drifting through a viscous liquid experience forces tending to clump them.

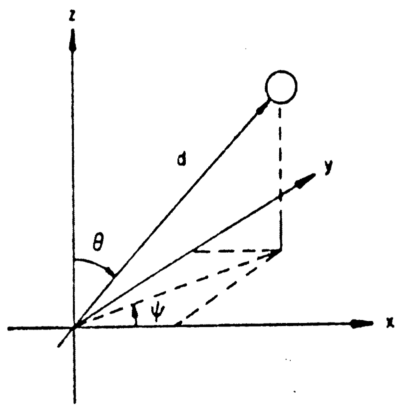

FIG. 2. Definition of coordinates for a sphere drifting through a viscous liquid.

arrangements of particles will include both horizontal and vertical neighbors, it may happen that the instability is quenched by stabilizing effects due to the additional particles above and below.

The present paper, which extends the analysis of the clumping instability to horizontal sheets, represents the first step in determining whether clumping can be expected in other arrangements besides the horizontal line. While originally intended as an exercise leading to the more important three-dimensional case, the results showed that the nearest neighbor approximation, which greatly simplifies the analysis, is severely limited in predicting the behavior of lattices in more than one dimension. Since the exclusion of the nearest neighbor approximation requires considerably more work in three dimensions, it is hoped that this paper will both justify the exclusion and furnish the basic equations and techniques for considering the effect of all neighbors in two and three dimensions. Actual predictions of stability, however, are given only for the two-dimensional horizontal sheet.

In the present work the analysis of a line of drifting particles will be extended to the case of an infinite horizontal sheet of particles falling through a viscous liquid.

Section II presents the general form of the equation of motion of a falling particle when influenced by another particle at an arbitrary position.

This result is used in Sec. III to study the behavior of the simplest horizontal lattice, a square lattice in which only the nearest neighbors affect the motion of any particle. Superposition of the effects of these four neighbors yields a difference equation which is solved to determine the growth rate and appearance of the clumping instability.

In Sec. IV the more complicated case of a square lattice with interactions among all of the particles is treated. Although the method is similar to the nearest neighbor case, the results show a striking difference: The most unstable disturbances have very long wavelengths, much longer than the interparticle spacing. This implies that the clumps which form in a horizontal sheet will be much larger than those expected by a simple generalization of the one-dimensional case.

Section V discusses some of the implications of this result.

II. THE FORCES ON TWO FALLING SPHERES

Since the instability develops in a lattice, it is often more convenient to express the interparticle forces in Cartesian coordinates rather than as the drag and line-of-centers vectors of Sec. I. Consider the forces acting on the particle at the center of the coordinate system shown in Fig. 2. Before the instability develops, all of the particles are falling at the same velocity, $-U$.

Once the drift velocity has been determined, the perturbations from the steady state can be analyzed by the solution of the equations of motion for a typical particle placed at the origin for convenience

$$6\pi\mu a \mathbf{u} = \mathbf{g} + \mathbf{f}. \tag{4}$$

In these equations, f represents the perturbation in the drag force when a neighboring particle is displaced from its original position and g represents the perturbation components of the line-of-centers force.

The perturbation drag force f can be derived from the expression for the drag force given earlier [Eq. (2)]. Consider a particle at the origin, with a neighbor at (x, y, z). The distance between the two particles can be written as

$$\mathbf{r} = x\hat{i}_x + y\hat{i}_y + z\hat{i}_z. \tag{5}$$

If the neighboring particle is displaced slightly to a new position

$$\mathbf{r} + \delta\mathbf{r},$$

the drag force is changed by an amount

$$\mathbf{f} = \nabla F_d \cdot \delta\mathbf{r} \tag{6}$$

so that the perturbation drag force is

$$\mathbf{f} = 6\pi\mu a U\left(\frac{3}{4}a\right)\frac{x\delta x + y\delta y + z\delta z}{R^3}, \tag{7}$$

where R

$$R = (x^2 + y^2 + z^2)^{1/2}. \tag{8}$$

Likewise, the perturbation line-of-centers force follows from Eq. (3) as

$$\frac{-2g_x}{9\pi\mu a^2 U} = \left(\frac{z}{R^3} - \frac{3x^2 z}{R^5}\right)\delta x - \left(\frac{3xyz}{R^5}\right)\delta y + \left(\frac{x}{R^3} - \frac{3xz^2}{R^5}\right)\delta z, \tag{9a}$$

$$\frac{-2g_y}{9\pi\mu a^2 U} = -\left(\frac{3xyz}{R^5}\right)\delta x + \left(\frac{z}{R^3} - \frac{3y^2 z}{R^5}\right)\delta y + \left(\frac{y}{R^3} - \frac{3yz^2}{R^5}\right)\delta z, \tag{9b}$$

$$\frac{-2g_z}{9\pi\mu a^2 U} = -\left(\frac{3xz^2}{R^5}\right)\delta x - \left(\frac{3yz^2}{R^5}\right)\delta y + \left(\frac{2z}{R^3} - \frac{3z^3}{R^5}\right)\delta z. \tag{9c}$$

Consider a square horizontal lattice of identical particles initially located at $(x, y, z) = (nd, pd, 0)$, where n and p are integers. After a perturbation the location of

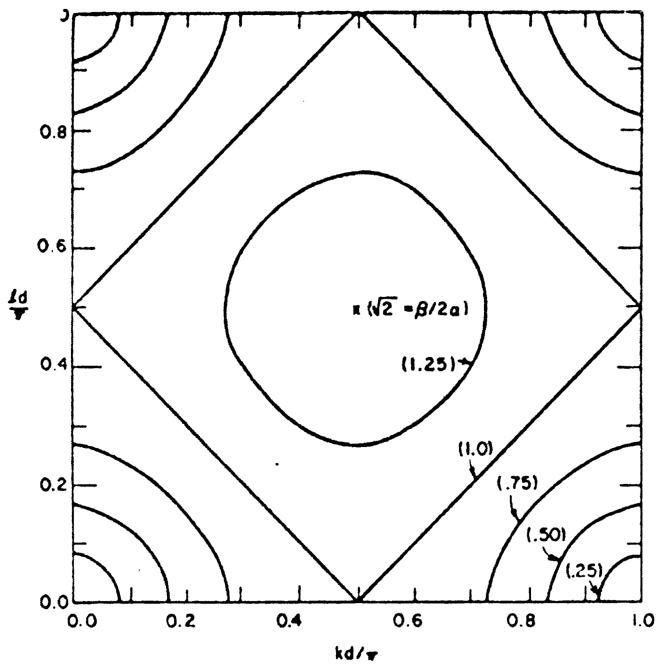

FIG. 3. Contours of growth rate in the wavenumber plans for a square lattice with only nearest neighbors interacting. Maximum growth occurs at $kd/\pi = ld/\pi = 1/2$.

the particle originally at $(nd, pd, 0)$ is given by

$$x = nd + \delta x_{np} ,\quad y = pd + \delta y_{np} ,\quad z = \delta z_{np}. \qquad (10)$$

The equation of motion for the particle initially at the origin (0, 0, 0) and affected only by its neighbor at $(nd, pd, 0)$ has the form

$$6\pi\mu a u = -\frac{9}{2}\frac{\pi\mu a^2 U}{d^2}\frac{n}{(n^2+p^2)^{3/2}}(\delta z_{np} - \delta z_{00}), \qquad (11a)$$

$$6\pi\mu a v = -\frac{9}{2}\frac{\pi\mu a^2 U}{d^2}\frac{p}{(n^2+p^2)^{3/2}}(\delta z_{np} - \delta z_{00}), \qquad (11b)$$

$$6\pi\mu a w = \frac{9}{2}\frac{\pi\mu a^2 U}{d^2}\left(\frac{n(\delta x_{np} - \delta x_{00}) + p(\delta x_{np} - \delta x_{00})}{(n^2+p^2)^{3/2}}\right). \qquad (11c)$$

III. THE STABILITY OF A HORIZONTAL LATTICE WITH NEAREST NEIGHBOR INTERACTIONS

When more than one neighbor is included, the perturbation forces for the additional neighbors are added to the right side of these equations. For instance, when the nearest neighbors at (0, 1), (0, − 1), (1, 0), and (− 1, 0) are considered, the equation for motion in the x direction becomes

$$6\pi\mu a u = -\tfrac{9}{2}(\pi\mu a^2 U/R^2)(\delta z_{10} - \delta z_{-10}). \qquad (12)$$

This equation, and the corresponding equations for the other velocities are differential-difference equations whose solution has the form

$$\delta x = \mathrm{Re}[\hat{x}\exp(\beta t)\exp(jknd)\exp(jlpd)]. \qquad (13)$$

With the additional definition

$$\alpha = \tfrac{3}{4}(aU/d^2), \qquad (14)$$

the equation for motion in the x direction becomes

$$\beta\hat{x} = -2j\alpha \sin kd\,\hat{z}\ . \qquad (15a)$$

Similarly, the equations for motion in the remaining two directions become

$$\beta\hat{y} = -2j\alpha \sin ld\,\hat{z} \qquad (15b)$$

$$\beta\hat{z} = 2j\alpha \sin kd\,\hat{x} + 2j\alpha \sin ld\,\hat{y}\ . \qquad (15c)$$

These equations have a solution only if the growth rate β takes on one of the eigenvalues

$$\beta = 0 \qquad (16)$$

$$\beta = \pm 2\alpha(\sin^2 kd + \sin^2 ld)^{1/2}. \qquad (17)$$

The existence of a positive eigenvalue for the growth rate indicates that the lattice is unstable, and small disturbances will grow and disrupt the regularity of the array. The growth rate varies with the wavelength of disturbances, and has its maximum value of

$$\beta_{max} = 2\sqrt{2}\,\alpha \qquad (18)$$

when $kd = \pm\pi/2$ and $ld = \pm\pi/2$. The growth rate at other wavelengths may be visualized by the Brillouin plot of Fig. 3 which shows contours of constant growth rate in the wavenumber planes. Figure 3 just shows the first quadrant of the entire Brillouin plot, but since the plot is symmetric about both the kd and the ld axes, there is no necessity of showing the remaining quadrants.

The Brillouin plot shows the most unstable wavelength for the instability, but does not give its form, which is needed in interpreting its effect on sedimentation. The form of instability is determined by substitution of the wavenumbers for maximum instability into the equations for the displacement amplitudes [Eq. (15)]. With $kd = ld = \pi/2$ and $\beta = 2\alpha\sqrt{2}$, these equations take the form

$$\sqrt{2}\,\hat{x} = -j\hat{z},\quad \sqrt{2}\,\hat{y} = -j\hat{z},\quad \sqrt{2}\,\hat{z} = j\hat{x} + j\hat{y}. \qquad (19)$$

From these equations, it is clear that the displacements in the x and y directions are equal, indicating that the particles move along straight lines when seen from above. The form of the instability, shown in Fig. 4, is obtained by substituting the wavelength for maximum instability and the relation between x and y components into the original assumed expression for the displacements

$$\delta x = \mathrm{Re}[\hat{x}\exp(\beta t)\exp(jknd)\exp(jlpd)] \qquad (20)$$

and evaluating at the original particle positions (n, p). After developing for some time, the instability would appear as alternate dense and light bands across the lattice.

So far, only the instability at $kd = ld = \pi/2$ has been discussed. Since the Brillouin plot is symmetric, there are three other wavelengths of maximum instability, corresponding to waves propagating at $\pm 45°$ to the lattice. When these waves are superposed, the particle displacements and density bands will appear as shown in Fig. 5, which resembles a pattern of dots. When only nearest neighbors interact, the dots will be spaced at twice the original spacing of the particles for the most unstable perturbation.

IV. THE SQUARE HORIZONTAL LATTICE WITH ALL NEIGHBORS INTERACTING

When more than the nearest neighbors affect the motion of the particle, the stability can still be analyzed as in Sec. III, using Eq. (4), (7), (9), and superposing the effects of all of the neighboring particles. Thus, the equation for motion in the x direction takes the form

$$u = -\alpha \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} n(n^2+p^2)^{-3/2}(\delta z_{np} - \delta z_{00}), \tag{21}$$

where the summation extends over all of the particles which affect the motion of the central particle. With the assumption of sinusoidal disturbances the equations of motion for the perturbations from the original lattice spacing may be written as

$$\begin{aligned} \beta\hat{x} &= -2j\alpha\hat{z}\, S(kd, ld), \\ \beta\hat{y} &= -2j\alpha\hat{z}\, S(ld, kd), \\ \beta\hat{z} &= 2j\alpha\hat{x}\, S(kd, ld) + 2j\alpha\hat{y}\, S(ld, kd), \end{aligned} \tag{22}$$

where

$$S(kd, ld) \equiv \frac{1}{2j} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} n(n^2+p^2)^{-3/2}[\exp(jknd)\exp(jlpd) - 1]. \tag{23}$$

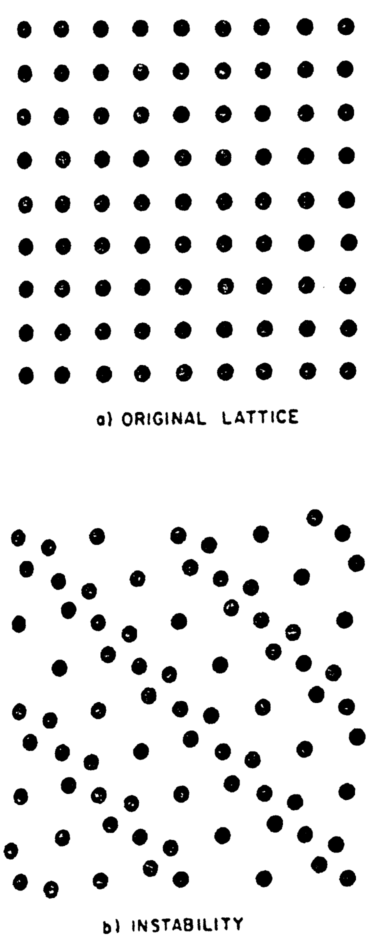

FIG. 4. Development of the instability for a disturbance in a single quadrant of the Brillouin plot.

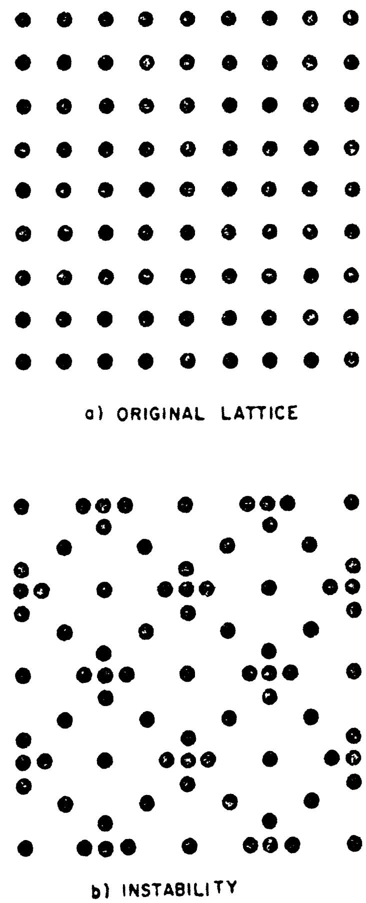

FIG. 5. Development of the instability for a disturbance with components in all four quadrants of the Brillouin plot.

These equations have a solution only if β assumes one of the eigenvalues

$$\beta = 0 \tag{24}$$

$$\beta = \pm 2\alpha\{[S(kd, ld)]^2 + [S(ld, kd)]^2\}^{1/2}. \tag{25}$$

This result is similar to that obtained when only nearest neighbors were considered. An important difference arises, however, when the Brillouin plots are calculated. The plot for all neighbors (Fig. 6) shows that the most unstable wavelengths do not occur near $\pi/2$ as with nearest neighbors, but near the origin of the plot, corresponding to very long wavelengths. This means that the instabilities which develop in a large horizontal array of particles will be much larger, and hence more visible, than would be expected by nearest neighbor theory.

V. DISCUSSION

The clumping that occurs on a horizontal sheet is similar, in many respects, to the clumping seen earlier on a falling line of particles. There is one difference, however, which has important implications in practice: In the horizontal sheet, the most unstable disturbances have infinite wavelengths. This implies that the clumps which develop spontaneously are likely to be much larger than the interparticle spacing. A sheet of particles with 10 μ spacing, for instance, will soon develop irregularities easily visible to the naked eye.

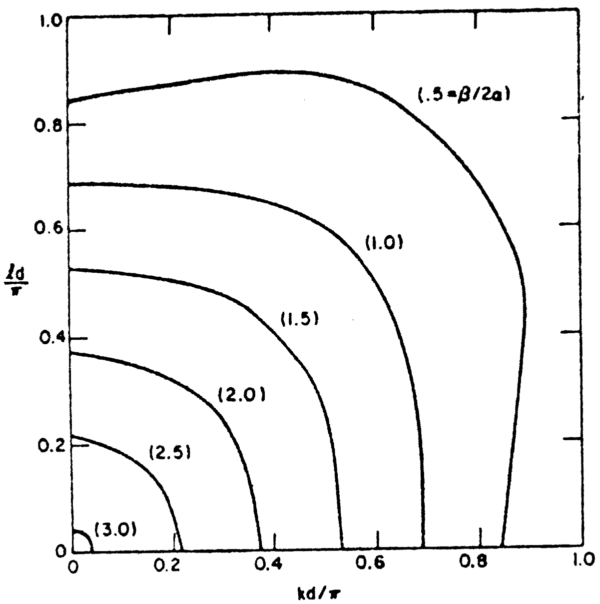

FIG. 6. Contours of growth rate with all neighbors interacting. Maximum growth occurs near the origin.

These large clumps can be predicted only when the nearest neighbor theory is extended to account for all interactions among particles. This distinct change in predicted behavior did not occur in the one-dimensional study when additional neighbors were considered. In two dimensions, however, there are many more neighbors at a given distance, and their combined effect will be greater than in the one-dimensional case. Clearly, the transition to a three-dimensional lattice will be accompanied by an even greater influence from the more distant neighbors, since there are even more neighbors in three dimensions than in two. Because of this, the temptation to generalize one- and two-dimensional results to a three-dimensional lattice should be resisted, even though (or especially since) the three-dimensional case is of great practical importance.

Although clumping always occurs in horizontal sheets, its growth may be so slow that it is not apparent under particular experimental conditions. If the particles are falling at a velocity U through a distance L, the growth in amplitude at the most unstable wavelength will be given by $\exp(3\pi aL/2d^2)$ from Eq. (25). It appears from this formula that the total growth is independent of the drift velocity, and depends only on the geometry. The quadratic factor of particle spacing in the exponent can be very important in determining whether the formation of clumps will occur rapidly enough to be noticeable. For example, if a lattice of particles with a 1 μ radius drifts a distance of 50 μ, the growth will be

$$\exp\left(\frac{3\pi}{2}\frac{(1)(50)}{(5)^2}\right) = 2.86\times 10^5$$

if the particles are spaced 5 μ apart (this corresponds approximately to a volume concentration of $\frac{4}{3}\pi a^3/d^3 = 33.5\times 10^{-3}$). The growth is only 1.1, however, with a 10 μ spacing (concentration $= 4.18\times 10^{-3}$). Thus, the growth of the clumps is increased by more than 10,000 times when the concentration of particles increases by a factor of 8. These results are, of course, based on linear theory, which is not likely to be valid for such large amounts of growth. It seems clear, however, that clumping becomes apparent at low concentrations, and that its magnitude increases very rapidly as the concentration is increased.

[1]J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics* (Prentice-Hall, Englewood Cliffs, N.J., 1965), Chap. 2.
[2]J. M. Crowley, J. Fluid Mech. 45, 151 (1971).
[3]H. Winet and T. Jahn, Biorheology 9, 87 (1972).
[4]B. Koglin, Chem. Ing. Tech. 44, 515 (1972).
[5]R. Vollheim, Maschinenbautechnik 21, 219 (1972).
[6]Y. A. Buyevich, J. Fluid Mech. 56, 313 (1972).
[7]F. A. Morrison, Chem. Eng. Sci. 28, 1115 (1973).