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# CENTRIFUGAL WAVES ON A FLEXIBLE CHAIN

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**Abstract**

A chain is attached at one end to a rigid rotating hub. Centrifugal force induces a nonuniform tension in the chain, which therefore supports transverse vibrations. Solution of the appropriate wave equation shows that the natural frequencies depend only on the rotational speed and the length of the chain relative to the hub diameter.

Experiments demonstrating the existence of these vibrations show that an initial impulsive disturbance reoccurs as an "echo" in which all the eigenmodes reinforce each other. This echo occurs even though the eigenfrequencies are not harmonically related. The explanation of the echo is found in the (almost) linear relation between eigenfrequency and mode number.

## § 1. Introduction

Rotating brushes are widely used in cleaning operations but the dynamics of their operation are not often discussed in the literature, although several interesting effects arise. One of these effects, "recovery oscillations", occurs when the bristles vibrate after striking an object.

In a slowly rotating, stiff brush these oscillations are essentially bending beam vibrations. In a rapidly rotating brush with limp, heavy bristles, however, the recovery oscillations are dominated by the centrifugal tension.

The present paper describes these centrifugal oscillations and discusses the conditions under which an initial distrubance may be recreated as an "echo".

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The radial dependence, $Y(R)$, is given by the equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[(1-r^2)\frac{\mathrm{d}y}{\mathrm{d}r}\right]+2\left(\frac{\omega}{\Omega}\right)^2 y=0 \tag{2.5}$$

where $r = R/L$. The solutions of this equation are Legendre functions

$$y = AP_l(r) + BQ_l(r) \tag{2.6}$$

in which

$$\frac{\omega}{\Omega}=\left[\frac{l(l+1)}{2}\right]^{\frac{1}{2}}. \tag{2.7}$$

The constants $A$ and $B$ are determined by the boundary conditions at either end of the chain. At the outer end, ($r = 1$), no forces are applied, so the transverse force must vanish

$$T(r)\left.\frac{\partial y}{\partial r}\right|_{r=1}=0. \tag{2.8}$$

The tension vanishes at the outer end, but the second Legendre function becomes infinite there. If the boundary condition is to be satisfied, the coefficient of the second Legendre function must vanish, so that the solution has the form

$$y = AP_l(r). \tag{2.9}$$

At the other end of the fiber, the displacement must vanish, or

$$P_l(a) = 0. \tag{2.10}$$

This condition determines the allowed values of $l$ for any point of attachment, $a$. This in turn gives the natural frequencies of the fiber through (2.7). These natural frequencies have been calculated for the range of attachment points ($0 \leq a \leq .95$) and plotted in Fig. 2. When the fiber is attached directly to the center of rotation ($a = 0$), the allowed values of $l$ are $= 1, 3, 5 \ldots$, corresponding to the first, third, fifth ... Legendre polynomials. As the point of attachment moves out toward the end of the fiber, the natural frequencies all increase, as shown in the Fig. The shape of the first few eigenmodes in shown in Fig. 3 for attachment at the center ($a = 0$). In all modes, the largest deflection occurs near the outer end of the fiber, where the centrifugal tension is smallest.

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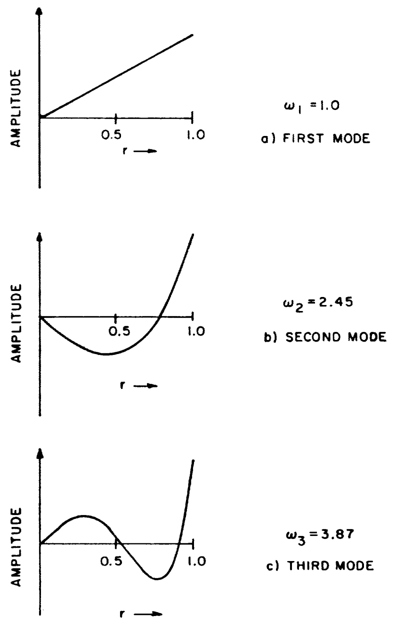


Fig. 3. The first three eigenmodes for centrifugal vibrations.

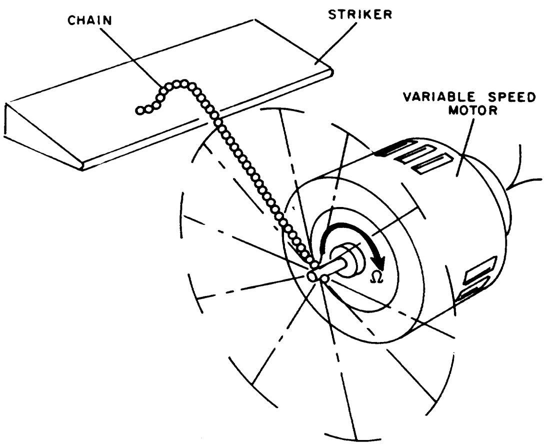


Fig. 4. A sketch of the experimental apparatus used for studies of the centrifugal vibrations.

are not harmonically related, as they are iú simple vibrating systems. When the experiment was performed, however, it happened that the original disturbance was almost exactly reconstructed at a point approximately opposite the initial disturbance.

Fig. 5 shows the response of the chain rotating in the clockwise direction and struck on its end at approximately 9 o'clock. This picture, taken with a stroboscope, shows the echo occurring at approximately 4 o'clock, after a complex series of vibrations. A complementary picture (Fig. 6), taken with a 30-second exposure, shows the echo position more clearly. It is apparent from this picture that the end of the chain is noticeable displaced only at the echo position, indicating that the amplitude of the disturbance is much larger there than anywhere else.

### § 4. The source of the echo

It was first suspected that the echo occurred because the striker excited one eigenmode very strongly. Calculated plots of the amplitude of the eigenmodes versus time (Fig. 7) disproved this conjecture, since none of the modes showed the behavior observed in the experiments. A striking characteristic of these plots, however, is the coincidence of all eigenmodes at a time corresponding to $\sim .6$ of one revolution. This indicates that even if the eigenfrequencies are not harmonically related, the initial disturbance can be recreated at some later time, forming an echo.

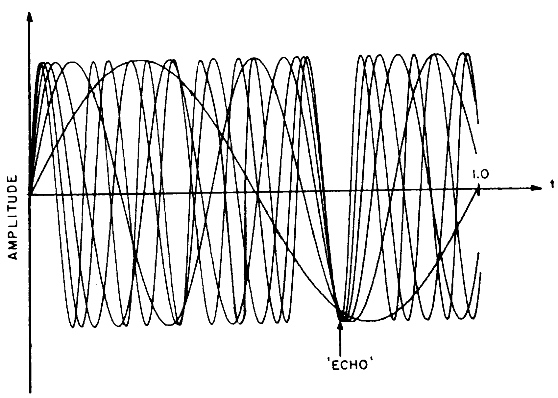


Fig. 7. Amplitude of the fiber end versus time for the first six centrifugal eigenmodes. Reinforcement of all modes occurs at the echo point.

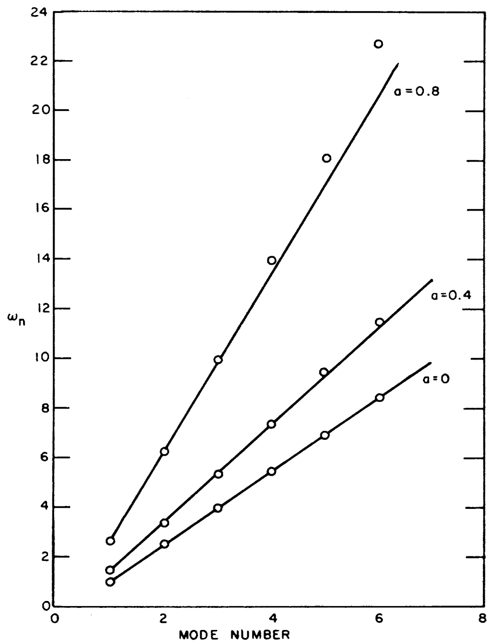


Fig. 8. The eigenfrequencies of the centrifugal vibrations increase linearly with the mode number if the fiber is attached close to the axis ($a \ll 1$).

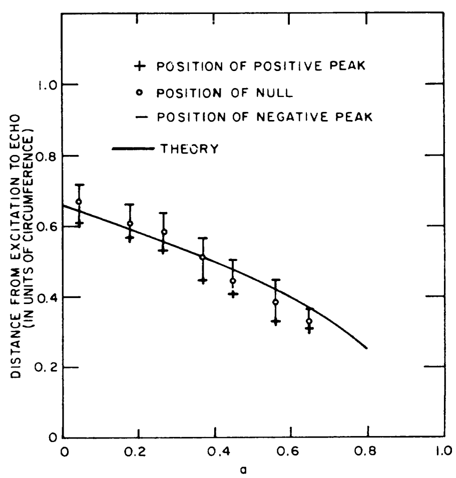


Fig. 9. Variation of the echo position as the attachment point is changed.

In the experiment described here, the echo occurs at a time determined by the point of attachment of the chain and its rotational speed.

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different values of attachment point, $a$. For low values of the parameter $a$, the plot is close to a straight line, and we should therefore expect an echo. For high values of $a$, the plot departs from linearity, and the echo should therefore be weak or absent.

This analysis indicates that the echo should occur at a time inversely proportional to the spacing between adjacent eigenfrequencies, which is determined only by the attachment point, $a$.

The position of the echo was measured on the experimental apparatus, and plotted versus "a" in Fig. 9. Since the echo consists of a rapid snap of the end of the chain, the actual position was bracketed by measurements of the points of maximum forward (+) and backward (—) displacement. Comparison of these results with the theory shows good agreement on all ranges of "$a$" for which the echo was present.

At higher values of the parameter $a$ ($> \sim .6$), the echo was weak or absent. This is to be expected, since the eigenfrequency spectrum becomes more nonlinear and reinforcement does not occur simultaneously for all modes.

## § 5. Summary

When a chain or flexible fiber is spun about its end, centrifugal tension leads to transverse oscillations which are described by Legendre's equation. The natural frequencies of these oscillations are given by

$$\frac{\omega_n}{\Omega} = \frac{\sqrt{l\,(l+1)}}{2}$$

where $l$ is determined by the attachment point of the chain through the transcendental equation

$$P(a) = 0.$$

Although the natural frequencies are not harmonically related, both theory and experiment show that an 'echo' can form at some time after an initial short disturbance. This echo, which reproduces the original pulse by superposition of eigenmodes, appears to be a possibility in any dynamic system in which the eigen frequency spectrum is close to a linear function of mode number, i.e.

$$\omega_n \simeq \alpha + \beta n.$$

This effect can be explained if the eigenfrequencies are represented by a Taylor series of the form

$$\omega_m = \alpha + \beta m + \gamma m^2 + \dots \tag{4.1}$$

where the mode number, $m$, can only assume integral values. A necessary condition for coincidence of two modes which are sinusoidal in time is that their phases be equal at the point of reinforcement. This condition may be written as

$$\omega_m t = \omega_n t [\text{mod } 2\pi] \tag{4.2}$$

where the symbol [mod] accounts for the periodicity of the phase. Substituting the expression for the eigenfrequencies into the phase condition yields

$$\beta(m - n)\, t + \gamma(m^2 - n^2)\, t + \dots = 0,\ 2\pi,\ 4\pi,\ \dots \tag{4.3}$$

or

$$t = \frac{2\pi}{\beta(m - n) + \gamma(m^2 - n^2) + \dots}\ (0, 1, 2, \dots). \tag{4.4}$$

The dependence of the coincidence time on the mode $m$ number indicates that different pairs of modes will coincide at different times, rendering the recreation of the original disturbance impossible. If the eigenfrequencies are linearly related to the mode number, however,

$$\gamma = \delta = \dots = 0 \tag{4.5}$$

the expression for coincidence time

$$t = \frac{2\pi}{\beta(m - n)} \tag{4.6}$$

depends only on the difference of the two mode numbers. Thus, the first and second modes will coincide at the same point that the second and third modes coincide, and all three modes must therefore coincide at the same point. A simple extension of this argument shows that modes coincide at the same point

$$t = 2\pi/\beta. \tag{4.7}$$

Thus, a strong echo will occur at $t = 2\pi/\beta$ if the eigenfrequencies are a linear function of the mode number. Fig. 8 shows the dependence of the centrifugal eigenfrequencies on mode number for

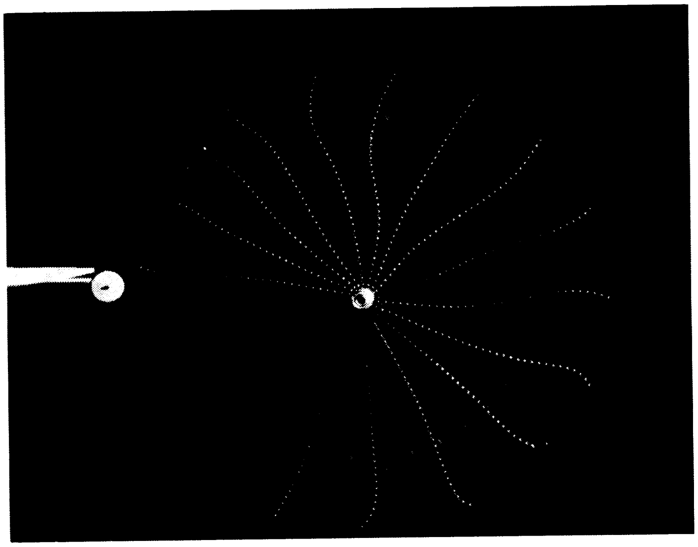

Fig. 5. A stroboscopic picture of the oscillations on the chain.

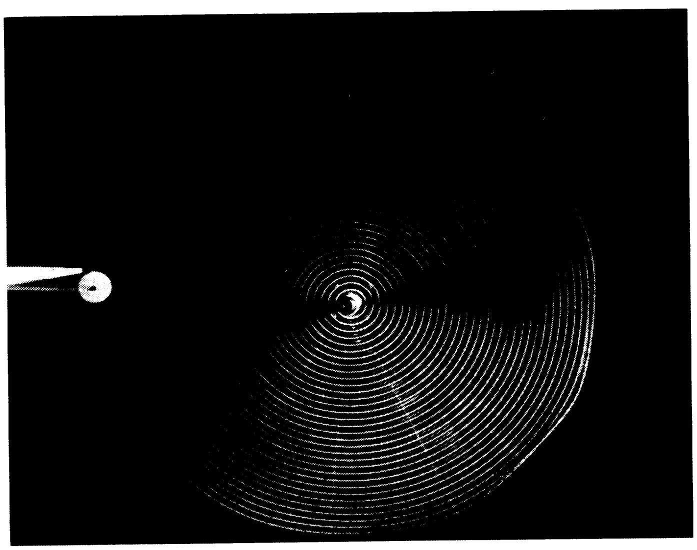

Fig. 6. A time exposure of the rotating chain, showing the position of the echo.

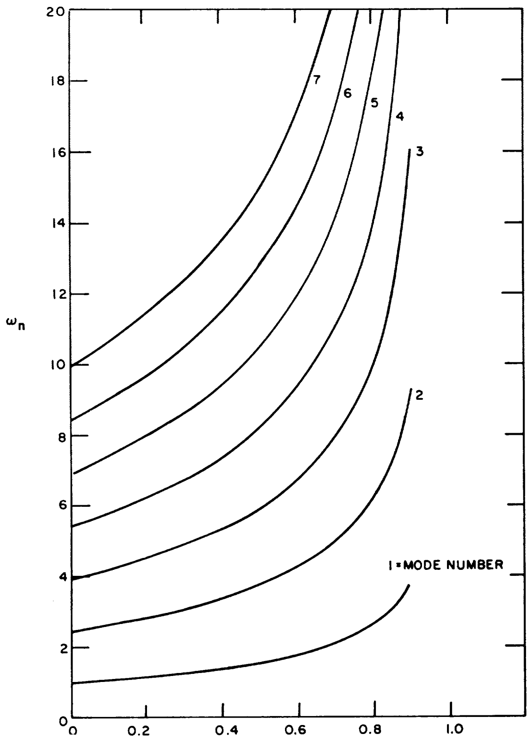


Fig. 2. The natural frequencies of the fiber as a function of attachment point.

### § 3. Experiments with the centrifugal wave

Several experiments were performed to determine whether these oscillations in fact exist, and whether they behave as predicted by the above analysis. The experimental apparatus consisted of a chain attached to the shaft of a variable speed motor as shown in Fig. 4. As the chain rotates, its end collides with a striker, whose position is adjustable. After hitting the striker, the chain oscillates as it rotates. This oscillation can be viewed under stroboscopic illumination or photographed.

When the chain is struck at the end, a broad spectrum of vibrations is excited, and it might be expected that the resulting vibrations of the chain will be incoherent because the eigenfrequencies

### § 2. The model

The centrifugal vibrations can be adequately described by a model with a single, limp fiber attached to a rotating hub, as shown in Fig. 1. This fiber will be called a "chain", to indicate that it has mass, but it is normally incapable of resisting bending. The chain can move in the azimuthal direction subject to a radial tension induced by centrifugal forces. The chain is attached to the hub at the point $R = R_a$, and extends to $R = L$. The chain has a density per unit length $\rho$, and rotates at the constant angular speed $\Omega$.

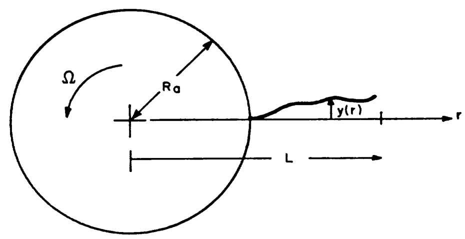


Fig. 1. A model of the brush fiber attached to a rotating hub.

The centrifugal tension in the fiber is determined by taking a radial force balance for a differential length of the chain yielding

$$\frac{dT}{dR} = -\rho\Omega^2 R. \tag{2.1}$$

Solution of this equation with the condition that the tension vanish at the end ($R = L$) gives the centrifugal tension as

$$T = \frac{\rho\Omega^2 L^2}{2}\left[1 - \left(\frac{R}{L}\right)^2\right]. \tag{2.2}$$

Following standard procedures, the wave equation for transverse vibrations of the chain is given by

$$\frac{\partial}{\partial R}\left[T(R)\,\frac{\partial y}{\partial R}\right] = \rho\,\frac{\partial^2 y}{\partial t^2}. \tag{2.3}$$

This equation has solutions of the form

$$y = y(R)\cos\omega t. \tag{2.4}$$