ELECTRICAL BREAKDOWN OF BIMOLECULAR LIPID MEMBRANES AS AN ELECTROMECHANICAL INSTABILITY

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ABSTRACT The bimolecular lipid membrane (BLM) is modeled as a bulk elastic layer subject to a compressive electric force caused by applied voltages. Analysis of this model shows that a compressive instability develops when the electric stress exceeds a critical value. This instability tends to crush the film and thus rupture it. The predicted breakdown voltage, when compared with measured values for phosphatidylcholine and cholesterol, shows fair agreement, considering the uncertainty in the estimate of elastic parameters.

INTRODUCTION

The BLM has often served as an experimental model of biological membranes (1–4), due to its similarity in many respects to the theoretical model suggested by Danielli and Davson. Both BLM's and membranes have similar thickness and capacitance, and both appear to be made up of amphipathic molecules in a bilayer arrangement. The first BLM's had very high electrical resistance and showed no action potential, but later preparations using additional components have been able to mimic natural membranes in both of these respects.

Several investigators have reported that BLM's rupture under an applied voltage which usually ranges from 100 to 1,000 mV. The mechanism of this breakdown has not received much attention, despite the fact that it is very important in experimental work.

In the work reported below, a mechanism for breakdown is proposed in which rupture occurs as a consequence of electrostatic compression of the elastic membrane. Since this mechanism depends on the mechanical properties of the membrane, it offers additional insight into the structure of the membrane, as well as a criterion for electric breakdown.

STABILITY OF AN ELASTIC CAPACITOR

The simplest mechanical model which includes the essential features of the proposed breakdown mechanism consists of a capacitor whose plates are separated by a uniform, isotropic elastic material. It differs from a film principally in having rigid, rather than flexible, conducting boundaries. Since the film surfaces are flat before breakdown develops, this simpler model will describe the film in the prebreakdown region, as well as furnishing a conceptual basis for the instability.

If a small steady pressure dp is applied to the ends of the elastic material, it will compress according to Hooke's Law

$$E(\mathrm{d}l/l) = -\mathrm{d}p, \tag{1}$$

where E is Young's modulus, and l is the length of the material. The total compression resulting from a finite pressure p is obtained by integrating Hooke's Law under the assumption that E is a constant,

$$E\int_{L}^{l} \mathrm{d}l/l = E \ln (l/L) = -p. \tag{2}$$

For the case in which the compression is small ($L - l \equiv \Delta \ll L$) this reduces to the familiar expression for the deflection of a spring

$$(E/L)\Delta = p, \tag{3}$$

where the spring constant per unit area is given by E/L.

For the spring separating the capacitor plates, the elastic force is balanced by the electric compressive tension,

$$-\epsilon V^2/2l^2 = E \ln (l/L), \tag{4}$$

where V is the voltage across the capacitor, and ϵ is its electrical permittivity.

When normalized to the original length of the spring, the equation takes the form

$$-\epsilon V^2/2EL^2 = (l/L)^2 \ln (l/L), \tag{5}$$

or in terms of the change in length Δ

$$-\epsilon V^2/2EL^2 = [1 - (\Delta/L)]^2 \ln [1 - (\Delta/L)]. \tag{6}$$

The numerical solution of this equation (Fig. 1) shows that the compression of the spring is initially proportional to the square of the voltage, and given by the approximate relation

$$\Delta/L \simeq \epsilon V^2/2EL^2. \tag{7}$$

There are at least two experiments reported in which a change in the thickness of the black film has been measured as a function of applied voltage. In both of these measurements, the compression was proportional to the square of the applied

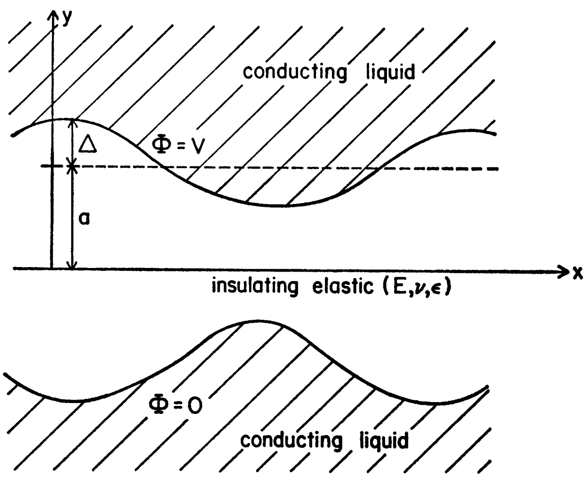

FIGURE 1 The membrane is modeled as an electrically insulating elastic layer surrounded by a conducting liquid.

voltage, as expected from this model (1, 4). Since the voltage, thickness, and dielectric constant (or capacitance) for these films can be estimated from other measurements, the compression equation can be used to estimate the Young's modulus of the membrane. Estimates for phosphatidylcholine and cholesterol membranes are reported in Appendix I.

At higher voltages, the deflection increases, approaching infinity at

$$\epsilon V^2/2EL^2 \simeq 0.18.$$

This is an indication of an electromechanical instability in which the mechanism may be described as follows. Consider a small displacement of the capacitor plates from the equilibrium position. If the displacement is compressive, the elastic force will increase, thus tending to return the plate to its initial position. As the plates approach each other, however, the electric pressure increases, according to its definition

$$-p_{\text{elec}} = \epsilon V^2/2l^2. \tag{8}$$

This increase in the electric compression tends to pull the plates even closer together. If the electric pressure dominates, the equilibrium is *unstable*, i.e. given a slight displacement from its equilibrium position, the plate tends to travel farther from equilibrium, in much the same way as a stick balanced on a fingertip tends to leave its equilibrium position.

The calculation of the condition for instability in the capacitor-spring model is not difficult; the result gives a minimum voltage in terms of the elastic modulus, the original spacing L and the dielectric constant. The result, as expected, is

$$\epsilon V^2/2EL^2 > \sim 0.18. \quad (9)$$

For the BLM, however, the opposite sides of the membrane are flexible and may deform, allowing the instability to occur at lower values of voltage. The effect of this flexibility is included in the analysis below, which results in a criterion predicting the magnitude of the applied voltage needed for instability (i.e., breakdown) of a thin elastic membrane.

STABILITY OF A FLEXIBLE MEMBRANE

The membrane is modeled by a layer of isotropic elastic material sandwiched between two semi-infinite, electrically conducting liquids maintained at a potential difference V. The entire voltage appears across the elastic layer, which is a relatively poor conductor of electricity, due to its high lipid content (Fig. 2). When the membrane is in mechanical equilibrium, its surfaces are flat, and the electric field is given by $V/2a$ where V is the potential drop across the membrane which has a thickness given by $2a$.

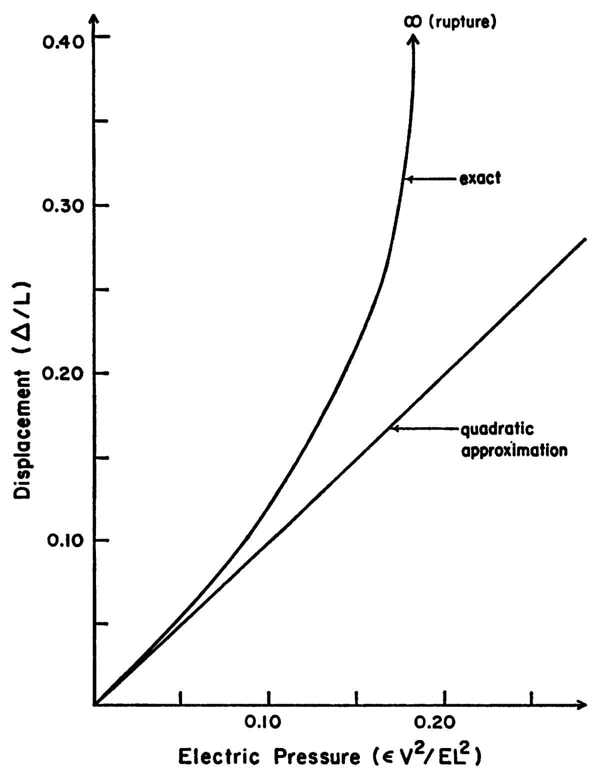

FIGURE 2 An applied electric field squeezes the membrane, and eventually ruptures it.

In a complete analysis of the stability of this membrane, the mechanical disturbances are expressed as a combination of sinusoidal displacements of the medium which would lead to deformations of the surface. These surface deformations distort the electric field within the membrane, thus changing the electric pressure at the surface. Both the electric and elastic effects are combined through a boundary condition at the surface of the membrane where the total stress must vanish.

This complete solution furnishes a great deal of information on the stability of the membrane, such as the growth rate of instabilities and the most unstable wavelength. In return for this information, a fairly accurate knowledge of the mechanical properties is demanded, so that the model should include the viscosity of both the membrane and the surrounding fluid, and other parameters. In addition, the evaluation of the resulting equations is somewhat cumbersome. Since the chief concern here is with the existence of instability, a somewhat simpler approach will be used.

In this approach, it is assumed that the instability develops as a monotonic growth of some small initial disturbance, as opposed to an oscillation of increasing amplitude. To determine instability, a small disturbance is assumed, and the resulting changes in the electric and elastic stresses are then calculated. If the net effect of the change in stress is to accelerate the departure from equilibrium, the system is unstable. If the two stress changes exactly balance each other, the membrane is at the critical point between stability and instability, where a slight increase in the voltage would cause a slow growth of the original disturbance. Since the growth near this neutral point is slow, effects such as viscosity which depend on the velocity of the growth may be neglected, and the instability problem reduces to finding the critical voltage at which the electric and elastic stress disturbances just cancel each other.

The simplest method of calculating these changes starts with the assumption of a sinusoidal deformation of the surface, of the form $\Delta \cos kx$. This represents a typical term in a Fourier expansion. This deformation, along with the equations governing the electric and elastic fields, determines the electric potential Φ and elastic displacement $\boldsymbol{\delta}$. At neutral equilibrium, these fields satisfy the shear and normal stress balance conditions at the surface of the membrane:

$$T_{xy}^{(\text{elas})} = 0, \tag{10}$$

$$T_{yy}^{(\text{elas})} + T_{yy}^{(\text{elec})} = 0. \tag{11}$$

The simultaneous solution of these two equations will then yield the critical voltage.

The calculation of the electric potential and elastic displacement fields is detailed in Appendix II. Use of the results obtained there in the stress balance equations gives the critical voltage for membrane breakdown as

$$\frac{(1 + \nu)\epsilon V_{cr}^2}{E(2a)^2} = 1, \tag{12}$$

where ν is the Poisson ratio for the elastic material of the membrane, E is its Young's modulus, ϵ is the electric permittivity, and V_{cr} is the critical voltage. The thickness $2a$ which appears in this equation is not the original thickness, since the membrane will have been compressed by the application of the electric field. From Eq. 5 the thickness at the critical potential is given by

$$(2a/L)^2 \ln (2a/L) = -\epsilon V_{cr}^2/2EL^2. \qquad (13)$$

Substitution of this value into the stability condition gives the condition

$$\epsilon V_{cr}^2/2EL^2 = \frac{e^{-1/(1+\nu)}}{2(1+\nu)}, \qquad (14)$$

for instability of the membrane. In this equation, L is the original thickness of the film.

The term on the right depends only on the Poisson's ratio of the elastic material. For real materials, $0 < \nu < \frac{1}{2}$. Evaluation of the right-hand side for the two extreme values ($\nu = 0, \frac{1}{2}$) gives 0.171 and 0.184, respectively. Since there is so little difference, the effect of ν on the instability may be ignored, and the stability criterion taken as

$$\epsilon V_{cr}^2/2EL^2 \simeq 0.18. \qquad (15)$$

It is interesting that this is essentially the same criterion obtained for the capacitor model (Eq. 9). This coincidence arises from the unusual result that all wavelengths of the flexible membrane become unstable at the same voltage. Since a flat plate is the limiting form of a long wavelength disturbance, its stability condition corresponds to that for long wavelength disturbances.

SOME APPLICATIONS TO EXPERIMENTS

The preceding analysis has provided an expression for the breakdown voltage for an elastic membrane, assuming that the breakdown occurs when compressive electrical forces overcome the elastic forces of the membrane. Since the breakdown voltages for some membranes have been measured, it should be possible to check the theory by comparing its predictions with the measured values if the thickness, dielectric constant, and Young's modulus are known.

Thickness

The thickness of black films (1) has been measured by light scattering techniques and by electron microscopy. The values obtained by these two methods indicate that most BLM's have thicknesses of 40–70 Å.

Dielectric Constant

This parameter is not measured directly. The dielectric constant is closely related to the capacitance of the bilayer, however, which is often reproducible among different experimenters, by

$$C_0 = \epsilon/L. \tag{16}$$

The ratio ϵ/L occurs in the breakdown criterion (Eq. 15), which may be rewritten to eliminate the dielectric constant, as

$$C_0 V_{cr}^2/2EL \simeq 0.18. \tag{17}$$

Young's Modulus

The description of the elastic properties of the membrane is the most difficult task in predicting electric breakdown. There are several factors which contribute to this difficulty.

First, the modulus calculated in Appendix I is based on experiments conducted at relatively small compressions, while the criterion is applied when the compression is large (~20%). Since there is no reason to expect that the membrane actually behaves as a linear elastic with constant modulus over such a wide range, the modulus obtained from low voltage experiments may be in error.

Second, there is some evidence that the elastic modulus of the membrane changes with time after a sudden change in applied voltage, perhaps due to interactions with the surrounding supports (5, 6). In general, the equilibrium membrane appears to be more yielding than the membrane undergoing rapid changes. This seems to point to a viscoelastic or hysteresis effect in the membrane, which has been ignored in the theory.

Finally, measurements of the apparent compressibility of the film seem to be affected by the composition of the surrounding medium, especially the electrolyte concentration (4). This indicates that the membrane is certainly more complex than the isotropic elastic material assumed in the theory.

Despite the uncertainty which exists in the elastic properties, it seems reasonable that the effective Young's modulus should be given by the results of Appendix I, at least within an order of magnitude. These estimates will now be used to compare the predictions of the theory with two BLM's which have known breakdown voltages. Table I shows the measured values of thickness, capacitance, Young's modulus, and breakdown voltage for BLM's composed of phosphatidylcholine and of cholesterol. The predicted breakdown voltage was computed for both of these films and is presented in the last row. These results are presented graphically in Fig. 3, which shows the breakdown voltage in millivolts vs. the value of the parameter EL/C_0 where E is given in dynes per square centimeter, L is in angstroms, and C_0 is in microfarads per square centimeter. The vertical error bars reflect the range in breakdown potentials reported in the literature. The horizontal error bars reflect the variation in compressibility reported by Rosen and Sutton (4) as the electrolyte concentration was varied. The actual uncertainty in the compressibility may well be larger than this; Haydon, for example, estimates the Young's modulus for phosphatidylcholine (1) as $\sim 3 \times 10^6$ dyn/cm^2. This value is almost 10 times as large as that used in Table I.

Despite this uncertainty, it appears from Fig. 3 that the experimental breakdown

TABLE I

COMPARISON OF THE PREDICTIONS OF THE THEORY WITH TWO BLM'S WHICH HAVE KNOWN BREAKDOWN VOLTAGES

Parameter	Phosphatidylcholine	Cholesterol
Thickness, L (Angstroms)	$\sim$50 (8)*	$\sim$50 (8)
Capacitance, C_0 (microfarads per square centimeter)	0.38 (9)	$\sim$0.9 (8)
Young's modulus, E (dynes per square centimeter)	$\sim 2.9 \times 10^5$	$\sim 2.7 \times 10^6$
Measured breakdown voltage (volts)	160–240 (10, 11)	400–600 (8)
Predicted breakdown voltage, V_{cr} (volts)	118	234

* Numbers in parentheses are reference numbers.

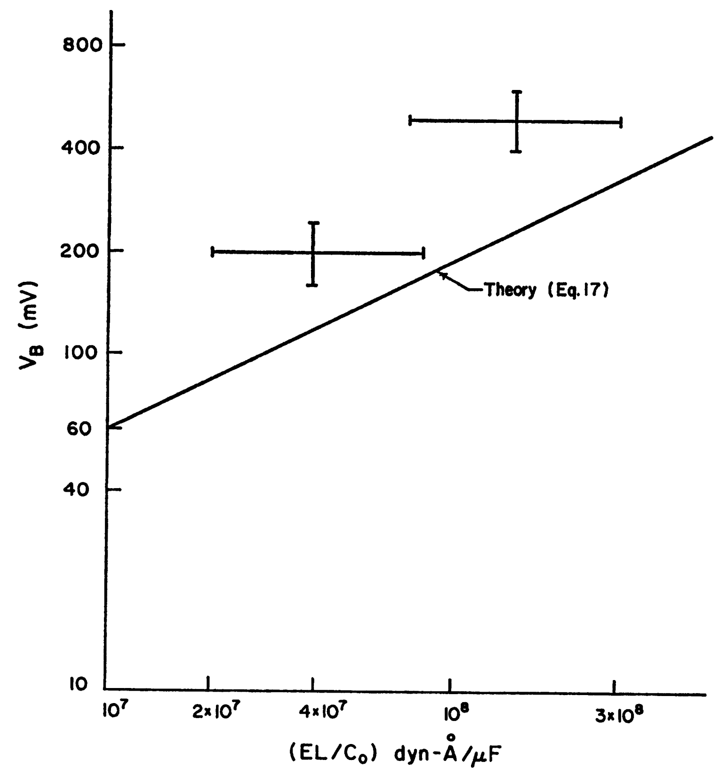

FIGURE 3 The breakdown voltage predicted by the present theory is close to that reported in the literature.

voltage is close to that predicted by the theory. In addition, the breakdown voltage for cholesterol is larger than that for phosphatidylcholine, as predicted by the criterion. Thus the breakdown criterion based on the electromechanical stability of the membrane seems to furnish at least a rough estimate of the measured breakdown voltage.

DISCUSSION

The theory presented above leads to a prediction for the breakdown voltage of an elastic membrane stressed by an electric field. Using this as a model of a black film, the predicted breakdown voltage was found to be close to the measured values for phosphatidylcholine and for cholesterol. In addition, the measured breakdown voltage for cholesterol is approximately three times that of phosphatidylcholine, as predicted. Thus there seems to be some evidence that the electroelastic instability accounts for the critical voltage of the black film.

The development of the instability once the critical voltage is exceeded cannot be described by the present theory, which is valid only for small disturbances from equilibrium. In practice, there are two possible outcomes. The first would be continued, accelerated growth of the disturbance. This would be expected if the electric compressive forces increase much faster than the elastic forces as the film is compressed. Eventually the disturbance would grow to the point at which it exceeded the thickness of the film, thus rupturing it. This would account for the breakdown of the BLM above a critical voltage.

The second possible outcome would be a decrease in growth as the instability develops, reaching a limit before the membrane is ruptured. This might occur if the voltage across the film could be discharged by ionization or diffusion through the film. Increased diffusion is certainly possible as the membrane becomes thinner, as are field-induced currents. This would not lead to rupture of the black film, but might furnish a mechanism by which permeability could be altered by the application of a voltage.

In the text, the difficulty in obtaining good values for the Young's modulus of the film was discussed. Although the actual elastic constants may not be those used in calculating the critical voltage, and the elastic itself may not be linear, it should not be concluded that the assumption that the membrane is a bulk elastic is not well founded. For example, it is often assumed that the elastic properties of a membrane may be represented by a surface tension which acts along the inner and outer surfaces, while the interior of the membrane is considered as a liquid. This model of the membrane has stability properties which are completely different from the bulk elastic model discussed in the present paper. As the length of the disturbance is made longer in the surface tension model, the breakdown voltage approaches zero, indicating that even a very small voltage would be sufficient to cause breakdown of the membrane (12). This occurs because the elastic restoring force of a stretched membrane is small if the membrane is very long. For the bulk elastic model used in the present work, however, the restoring force is distributed along the entire length of the membrane and is therefore able to resist disturbances of any length.

Experimentally, the membrane is able to support relatively large potentials before breakdown, which implies that the elastic properties of the membrane cannot be represented as a surface tension, at least when film stability is under discussion. It is occasionally suggested that the breakdown of the membrane could be an avalanche process similar to that which causes sparking in air breakdown. In support of this suggestion is the fact that the electric field strength in the membrane at breakdown is on the order of 20×10^6 V/m, which is comparable with the breakdown voltage for most oils (13). In reality, the avalanche process is extremely unlikely in the membrane, since it requires that the total voltage across the dielectric exceed the ionization potential of the material. When the applied potentials are measured in thousands of volts, as in air breakdown, this last requirement is no real restriction. In a membrane, however, the potentials are measured in millivolts, and the charge carriers can not acquire enough energy in drifting across the membrane to ionize any of the surrounding molecules through collisions. Since the creation of additional carriers through collisions is the crucial step in the avalanche process, we may rule out avalanching as a source of breakdown in membranes.

There are other breakdown mechanisms in liquids but these effects usually occur at higher field strengths than those obtained in black film or membrane experiments. Thus it appears that electromechanical instability offers the most likely explanation for the electric breakdown of the black film.

Finally, it should be pointed out that the application of the breakdown criterion to experimental membranes, which was reported above, can only be an estimate, since the breakdown voltages and elastic constants did not result from measurements on the same membrane. In practice, there is often considerable variation in the stability of different membranes. Work is now in progress to obtain elastic constants and breakdown voltages for the same membrane.

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APPENDIX I

Estimation of Young's Modulus from Experiments

The estimates presented here are based on measurements of the change in capacitance of a BLM as the steady voltage across the film is increased. In these measurements, reported by Rosen and Sutton (4), the capacitance increased linearly with the square of the applied voltage, as expected from the discussion above.

The capacitance of the film may be expressed as

$$C = \epsilon/l \qquad (A\ 1)$$

if the film is considered as a flat insulating layer. If the thickness of the layer changes by Δ, the capacitance will change as

$$\Delta C/C_0 \simeq -\Delta/L. \tag{A 2}$$

Using the expression of Eq. 7 which relates the change in thickness to the applied voltage and the elastic parameters, the Young's modulus can be estimated as

$$E = C_0/2L\left(\frac{\Delta C/C_0}{V^2}\right)^{-1}. \tag{A 3}$$

Table II shows the measured values of the membrane parameters, along with the implied value of Young's modulus, calculated using Eq. A 3 above. For phosphatidylcholine $\Delta C/C_0$ was read from Fig. 3 of Rosen and Sutton's paper at an applied voltage of 0.01 $(V)^2$.

The slope for cholesterol was obtained from their Fig. 4, by selecting an electrolyte concentration which gave the same change in capacitance at 50 mV as for the phosphatidylcholine film obtained in Fig. 3. At this concentration ($\sim$30 meq/liter), $\Delta C/C_0$ for cholesterol was approximately one-fourth of the corresponding change for phosphatidylcholine. These values for the modulus should only be considered as estimates since the results for this type of experiment may vary for different tests. Rosen and Sutton report, for example, that the change in capacitance may vary by a factor of three as the electrolyte concentration of the aqueous media is changed, while the cholesterol used in their experiment was of uncertain vintage.

TABLE II

MEASURED VALUES OF MEMBRANE PARAMETERS AND IMPLIED VALUE OF YOUNG'S MODULUS CALCULATED USING EQ. A 3

Parameter	Phosphatidylcholine	Cholesterol
Capacitance C_0 (microfarads per square centimeter)	$\sim$0.38 (9)*	$\sim$0.9 (8)
Thickness L (angstroms)	$\sim$50 (8)	$\sim$50 (8)
Slope $(\text{volts})^{-2}$	13	3.25
Young's modulus E (dynes per square centimeter)	2.9×10^5	2.7×10^6

* Numbers in parentheses are reference numbers.

APPENDIX II

Calculation of Electric and Elastic Stresses in the Deformed Membrane

As a result of the initial disturbance the surface of the membrane is displaced from its equilibrium position as shown in Fig. 2. The position of the upper (+) and lower (−) surfaces is given as

$$y = \pm(a + \Delta \cos kx). \tag{A 4}$$

The disturbance is therefore assumed to be both symmetric and sinusoidal. Antisymmetric disturbances may occur, but are neglected in this study of breakdown since they have little effect on the thickness of the membrane. Nonsinusoidal disturbances may also occur, but they can always be expressed via Fourier techniques as combinations of sinusoidal disturbances. Thus this assumed deformation represents all small disturbances which change the thickness of the membrane. The electric potential inside the deformed membrane is given by

$$\Phi(x, y) = \frac{(a + y)}{2a} V + \phi(x, y). \tag{A 5}$$

The perturbation field satisfies the electrostatic equation

$$\nabla^2 \phi = 0, \tag{A 6}$$

which has the solution

$$\phi = (C_1 \sinh ky + C_2 \cosh ky) \cos kx \tag{A 7}$$

in rectangular geometry. The term $C_2 \cosh ky$, which corresponds to antisymmetric deformations, will be neglected.

The constant C_1 is determined by the boundary condition at the surface of the membrane. For a conducting boundary, the potential has the constant value V

$$V = \left[\frac{V(a + y)}{2a} + C_1 \sinh ky \cos kx\right] \tag{A 8}$$

at $y = a + \Delta \cos kx$. For small disturbances ($\Delta/a \ll 1$), this gives

$$C_1 = -V\Delta/(2a \sinh ka). \tag{A 9}$$

In the stress balance at the surface of the marginally stable membrane, only the normal component of the electric stress tensor is needed if the surface is only slightly deformed ($\Delta/a \ll 1$). This component is

$$T_{yy} = (\epsilon/2)[(\partial\Phi/\partial y)^2 - (\partial\Phi/\partial x)^2], \tag{A 10}$$

where ϵ is the electric permittivity of the membrane. With the solution for Φ, this stress takes on the approximate value

$$T_{yy}\big|_{y=a+\Delta \cos kx} = \frac{\epsilon V^2}{2(2a)^2} - \frac{kV^2\Delta}{(2a)^2} \coth ka \cos kx \tag{A 11}$$

at the surface of the membrane. In this expression, the higher order terms in Δ have been neglected. The first term represents the static pressure which exists in the equilibrium state, while the second term accounts for the change in pressure caused by the deformation.

The static deformation of an isotropic elastic material is given by

$$\bar{\delta} = \nabla\psi + \nabla \times \bar{A}, \tag{A 12}$$

where

$$\nabla^2\psi = 0, \tag{A 13}$$

and

$$\nabla \times \nabla \times \bar{A} = 0. \tag{A 14}$$

The solution of these equations gives ψ and $\bar{A}$ as

$$\psi = C_3 \cosh \alpha y \cos kx, \tag{A 15}$$

and

$$A_z = C_4 \sinh \beta y \sin kx, \tag{A 16}$$

for the symmetric mode. If the deformations were dynamic (as in elastic vibrations), the constants α and β would be unequal. For static deformations, however, they reduce to the same value, $\alpha \rightarrow k$ and $\beta \rightarrow k$, and the solution becomes degenerate. A simple way to handle this degeneracy is to keep α and β unequal until the stress has been calculated, and then let them both approach their limiting value k.

The displacement of the elastic medium is given by

$$\delta_x = -kC_3 \cosh \alpha y \sin kx + \beta C_4 \cosh \beta y \sin kx, \tag{A 17}$$

$$\delta_y = \alpha C_3 \sinh \alpha y \cos kx - kC_4 \sinh \beta y \cos kx.$$

The constants C_3 and C_4 are determined by the boundary conditions on shear and normal stresses at the interface. If the surrounding fluid is a conducting liquid, the shear stress will vanish for the neutrally stable displacements,

$$T_{xy}^{(\text{elas})}\Big|_{y\simeq\pm a} = G\left(\frac{\partial \delta_x}{\partial y} + \frac{\partial \delta_y}{\partial x}\right)\Bigg|_{y\simeq\pm a} = 0, \tag{A 18}$$

where $G = E/2(1 + \nu)$ is an elastic constant which depends on the Young's modulus E and the Poisson ratio ν. Using the expressions for δ_x and δ_y, the shear condition may be written as

$$-2k\alpha \sinh \alpha a\, C_3 + (\beta^2 + k^2) \sinh \beta a\, C_4 = 0. \tag{A 19}$$

The normal elastic stress at the surface is

$$T_{yy}^{(\text{elas})} = (2G + \lambda)(\partial \delta y/\partial y) + \lambda(\partial \delta x/\partial x), \tag{A 20}$$

where

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}.$$

Using the expressions for δ_x and δ_y, this becomes

$$T_{yy}^{(\text{elas})} = [(2G + \lambda)\alpha^2 - \lambda k^2]C_3 \cosh \alpha a \cos kx$$

$$-2k\beta G\, C_4 \cosh \beta a \cos kx. \tag{A 21}$$

At the point of marginal stability, the sum of the electric and elastic normal stresses will vanish, or

$$\left\{[(2G + \lambda)\alpha^2 - \lambda k^2] \cosh \alpha a - \frac{\alpha\epsilon k V^2}{(2a)^2} \coth ka \sinh \alpha a\right\} C_3 + \left\{2k\beta G \cosh \beta a - \frac{k^2 \epsilon V^2}{(2a)^2} \coth ka \sinh \beta a\right\} C_4 = 0. \quad (\text{A } 22)$$

In writing this equation, only the change in stress due to the surface deformation has been used. In the electric pressure term, which comes from Eq. A 11, the amplitude of the surface displacement Δ has been replaced by the y component of the elastic displacement at the surface from Eq. A 17. Simultaneous solution of the two stress equations, and evaluation of the limit $\alpha \rightarrow k$, $\beta \rightarrow k$ gives the relatively simple condition for stability of the interface

$$\frac{(1 + \nu)\epsilon V^2}{E(2a)^2} < 1. \quad (\text{A } 23)$$

This result is much simpler than most stability criteria for distributed systems, since it is independent of the wavelength of the disturbance. When the critical voltage is exceeded, all disturbances become unstable. In practice, of course, some wavelengths will grow faster than others, but this cannot be determined from the present analysis. Experience with other instabilities suggests, however, that the most unstable wavelengths will be related to the thickness of the membrane.

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