OOCP

Reprinted from *Printed in U. S. A.*

THE PHYSICS OF FLUIDS VOLUME 11, NUMBER 10 OCTOBER 1968

Growth of Waves on an Accelerated Jet

JOSEPH M. CROWLEY

Max Planck Institut für Strömungsforschung, Göttingen, Germany

and

*Charged Particle Research Laboratory, Department of Electrical Engineering, University of Illinois Urbana, Illinois**

(Received 20 February 1968; final manuscript received 25 June 1968)

An electric field normal to a conducting liquid jet causes spatial instability of the kink mode. The growth rate of this instability can be calculated from a dispersion relation only if the jet velocity is constant. In this paper, the growth of waves on a falling jet is studied under the assumption that the fractional change in the velocity is small, and several approximate dispersion relations for the limit of very fast and very slow growth are given.

I. INTRODUCTION

Spatially growing waves occur in such diverse physical systems as the laminar boundary layer,[1] the electron beam amplifier,[2] the liquid jet,[3] and the thin liquid film.[4] The assumption that the parameters of the system are constant in the direction of wave propagation simplifies the theoretical study of these waves to the point that the growth rate is then a constant determined by the dispersion relation in terms of the applied frequency and transverse parameters of the system. This simplification of theory, however, leads to difficulty in evaluating experimental results because the physical systems often change in the wave direction. For example, the thickness of the laminar boundary layer increases in the direction of flow, as does the velocity of a falling liquid jet. In such cases the problem of bridging the gap between theoretical predictions and experimental results must be solved.

The first and most straightforward approach to this problem is the reduction of the inhomogeneity in the physical system by careful design of the experiment. If sufficient reduction cannot be achieved, a correction to the theoretical results must then be sought to take account of the remaining discrepancy. The latter approach will be discussed in the remainder of this paper in terms of the transverse kink waves on a liquid jet, a system which has recently enjoyed great experimental popularity.[3,5–8]

* Present address.

[1] H. Schlichting, *Grenzschicht-Theorie* (Verlag G. Braun, Karlsruhe, 1958), Chap. 16.

[2] J. R. Pierce, *Travelling Wave Tubes* (D. Van Nostrand Company, Inc., New York, 1950).

[3] J. R. Melcher, *Field Coupled Surface Waves* (The M.I.T. Press, Cambridge, Massachussetts, 1963), Chap. 6.

[4] T. B. Benjamin, J. Fluid Mech. 2, 554 (1967).

II. DESCRIPTION OF THE SYSTEM

Figure 1 shows a liquid jet of radius $R(x)$, density ρ, and surface tension T. The jet falls with a velocity $V(x)$ which increases under the influence of gravity from the entrance to the exit of the length under consideration ($0 < x < L$). A concentric electrode of radius d at a potential Φ_0 surrounds the grounded jet, thus setting up an electric field in the radial direction. Under the influence of this field a transverse disturbance of the jet $\delta(x, t)$ will grow in space as it propagates downstream.

To simplify the analysis, we shall assume that the wavelength of the disturbance is very long compared with the diameters of the jet and the electrode. Under these conditions, the equation of motion can be derived from the transverse force balance on a

[5] U. Ingard and D. S. Wiley, Phys. Fluids 5, 1500 (1962).

[6] S. Middleman and J. Gavis, Phys. Fluids 8, 222 (1965).

[7] J. M. Crowley, Phys. Fluids 8, 1668 (1965).

[8] F. D. Ketterer, Ph.D. thesis, Massachusetts Institute of Technology (1965).

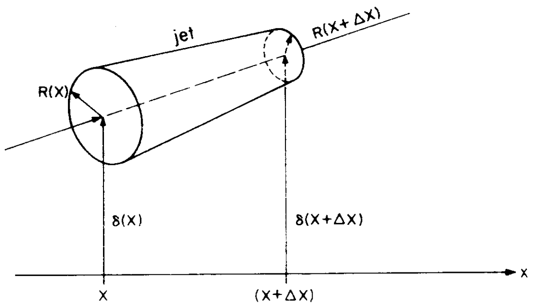

FIG. 2. A force balance on a differential length of the jet gives the equation of motion in the long wave limit.

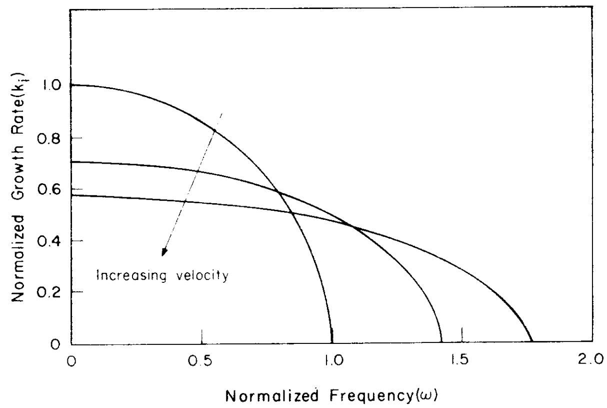

FIG. 3. As the velocity increases, the growth rate at low frequencies decreases, while the growth at high frequencies increases.

where $\beta = 2gL/U_0^2$ measures the effect of the gravitational acceleration over the length of interest. By conservation of mass, the radius of the jet is

$$\frac{R(x)}{R(x=0)} = U(x)^{-\frac{1}{2}}. \tag{8}$$

The boundary conditions are both applied at the point of excitation if $\alpha^2 < 1$ (supercapillary jet). For an exciter which applies a force over a short length of the jet, the normalized boundary conditions are very nearly of the form[7]

$$\hat{\delta}(0) = 0, \tag{9a}$$

$$\frac{d\hat{\delta}}{dx}(0) = 1. \tag{9b}$$

III. GROWTH ON THE UNACCELERATED JET

Any disturbance on a supercapillary jet moving at a constant velocity ($\beta = 0$) is composed of terms of the form[3]

$$\exp(\pm k_i x) \exp i(\omega t - k_r x),$$

where

$$k_r = \frac{\omega}{1-\alpha^2}, \tag{10a}$$

$$k_i = \frac{[N(1-\alpha^2) - (\alpha\omega)^2]^{\frac{1}{2}}}{1-\alpha^2}. \tag{10b}$$

This represents waves propagating in the direction of flow with a phase velocity $c_r = \omega/k_r$. The amplitude of these waves grows or decays according as the sign of k_i is positive or negative. The quantity k_i is called the growth constant, and its study is the chief goal of experimental investigations of growing waves.[3,7,13,14]

The growth rate depends on the velocity of the jet, which appears in the quantities N, α, and ω. This dependence is illustrated in Fig. 3, where k_i is plotted versus ω for various values of velocity while keeping all other parameters constant. At low frequencies, increasing the velocity lowers the growth rate because the disturbance is swept downstream faster. At higher frequencies, in the vicinity of cutoff, another effect dominates. Here the growth rate is proportional to wavelength and, since increasing the velocity increases the wavelength, the growth rate increases.

IV. GROWTH ON THE ACCELERATED JET

In the usual method of measuring the growth rate, a force is applied to the jet at some upstream point, and the magnitude of the resulting disturbance is measured at two points along the jet. The fractional difference between these two measurements represents the growth of the wave over the distance separating the two points. If the results of this measurement are to be compared to theoretical predictions based on constant jet velocity, the fractional change in velocity between the two points should be kept as small as possible. One possibility is to shorten the length under consideration. As the section becomes shorter, however, the difference in amplitudes becomes smaller, and consequently more difficult to measure accurately. The fractional change in velocity can also be decreased by increasing the velocity, but since the growth rate of the waves is inversely proportional to the velocity, the growth over the measurement length is again decreased. In addition, the jet will break up into droplets at a shorter distance from the nozzle if the velocity is too high.[15,16] Thus, in practical work, the theory

[13] G. B. Schubauer and H. K. Skramstad, NACA Report No. 909 (1943).
[14] A. M. Binnie, J. Fluid Mech. 2, 551 (1967).
[15] A. Haenlein, Forschung 2, 139 (1931).
[16] C. Weber, Z. Angew. Math. Mech. 11, 136 (1931).

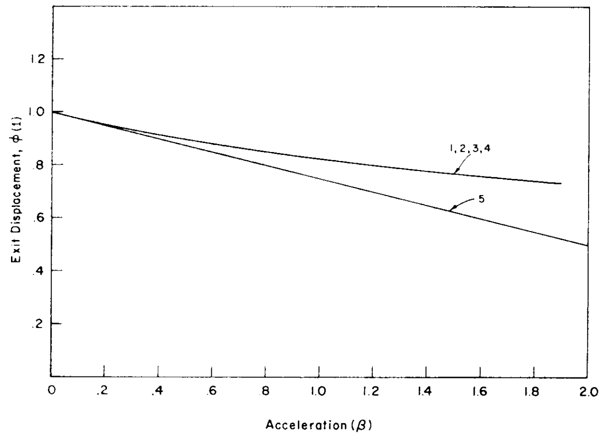

FIG. 4. Comparison of solutions with no wave growth: 1. numerical solutions, 2. $(k_i)_{ea}$, 3. $(k_i)_{eb}$, 4. $(k_i)_{ec}$, 5. perturbation approximation ($N = 0$, $\omega = 0$, $R/d = 0.322$, $\alpha^2 = 0.01$).

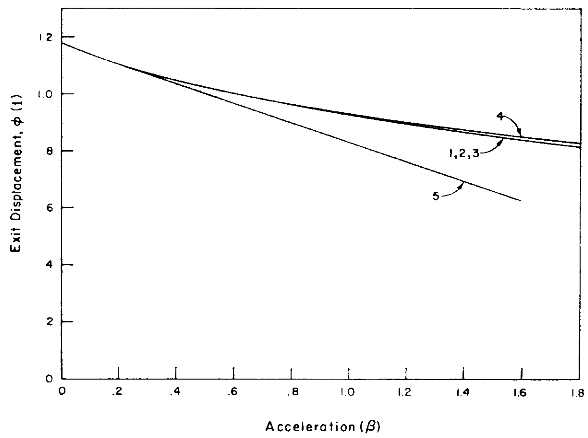

FIG. 5. Comparison of solutions for moderate growth ($N = 1$, $\omega = 0$, $R/d = 0.322$, $\alpha^2 = 0.01$).

and compared to the numerical solution of the equation of motion. They are

$$(k_i)_{ea} = \int_0^1 k_i(x)\,dx, \tag{17}$$

$$(k_i)_{eb} = k_i(x = \tfrac{1}{2}), \tag{18}$$

$$(k_i)_{ec} = \tfrac{1}{2}[k_i(0) + k_i(1)], \tag{19}$$

where $k_i(x)$ is given by

$$k_i(x) = \frac{\{N(x)[U^2(x) - \alpha^2(x)] - \omega^2\alpha^2(x)\}^{\frac{1}{2}}}{U^2(x) - \alpha^2(x)}.$$

The first approximation represents the mean growth rate by an integral over the length of jet under consideration. The second and third expressions may be considered as approximations to this integral, valid when the growth rate does not change appreciably over the length of interest.

The approximate growth rate technique is not restricted to the long wave limit considered here in which the disturbance is completely represented by a differential equation; it can be used whenever a dispersion relation for the corresponding homogeneous case can be formulated. The only restriction is that the growth rate of the disturbance is much larger than the rate of change of the physical parameters (velocity, radius, etc.) which determine the growth rate.

For the accelerated jet, an additional correction must be made, since the transverse wave amplitude is affected by gravity even in the absence of instability or wave motion ($k_r = k_i = 0$). The envelope will have a parabolic shape, characteristic of a falling body with an initial horizontal velocity which is given by

$$\hat{\delta} = \frac{(1 + \beta x)^{\frac{1}{2}} - 1}{\frac{1}{2}\beta} \tag{20}$$

neglecting the effect of surface tension. The approximate expression for wave growth must, therefore, be multiplied by this factor to obtain the complete expression for the displacement amplitude of the jet,

$$\hat{\delta} = \left(\frac{\sinh (k_i)_{eff}x}{(k_i)_{eff}x}\right)\left(\frac{(1 + \beta x)^{\frac{1}{2}} - 1}{\frac{1}{2}\beta}\right), \tag{21}$$

where $(k_i)_{eff}$ is one of the three effective growth rates given in Eqs. (17)–(19).

VIII. COMPARISON OF DIFFERENT EXPRESSIONS

These three approaches are compared in Figs. 4–7 for several different operating points. The figures show the displacement amplitude at the end of the section of jet ($x = 1$) as a function of total change in velocity over this length. Figure 4 represents a zero-frequency case in which the electric field (and hence the growth) is absent. Since $k_i = 0$ under these conditions, only gravity affects the amplitude of the disturbance. All three of the approximations are identical, of course, and they also agree quite well with the numerical solution. This offers evidence that the parabolic term [Eq. (20)] accounts for the gravity deformation satisfactorily.

The perturbation result, which is linear in β, cannot follow the nonlinear exact solution, but its slope at low β is equal to that of the other results, indicating its validity.

Figures 5 and 6, which show the zero-frequency response for $N = 1$ and $N = 25$ (growth rates $k_i \simeq 1$ and $k_i \simeq 5$, respectively) indicate that the

the study of waves in nonuniform media, such as boundary layers.[13] It should be stressed that it is not valid at marginal stability, since the growth rate in this situation is small and the more general approach given above should be used.

An analysis of Eq. (22) shows that the error introduced into the growth constant by neglecting the second term is of the order $1/k_i x$, and is due principally to the decaying wave excited at the origin. Thus, the total growth from the exciter to the test section should be large for good accuracy.

XI. SUMMARY

The magnitude of a growing disturbance on an accelerated liquid jet is approximately given by

$$\delta = \left(\frac{\sinh (k_i)_{\text{eff}} x}{(k_i)_{\text{eff}} x}\right)\left(\frac{(1 - \beta x)^{\frac{1}{2}} - 1}{\frac{1}{2}\beta}\right), \qquad (21)$$

where $(k_i)_{\text{eff}}$ is an average growth rate over the length of the jet. If $(k_i)_{\text{eff}}$ is calculated by an integral mean

$$(k_i)_{ea} = \int_0^1 k_i(x)\, dx, \qquad (17)$$

the error is less than 1% for $\alpha \ll 1$, $N \leq 25$, $\beta \leq 2$, $\omega \leq 0.9\omega_{\text{cutoff}}$ while for the simpler expression

$$(k_i)_{eb} = k_i(x = \tfrac{1}{2}), \qquad (18)$$

the error over the same range is less than 10%.

The perturbation expression (Eq. 16) gives greater than 10% error for $\beta > 0.3$ over the same range.

When the section of jet under consideration is far from the excitation, the growing wave dominates all other disturbances and the wave envelope is given by the expression

$$\hat{\delta}(x + \Delta x) = \hat{\delta}(x) e \int_x^{x+\Delta x} k_i(x)\, dx, \qquad (25)$$

where $k_i(x)$ is calculated using the local parameters. The error in this expression depends on the distance between the excitation and the test section and is approximately given by

$$\text{fractional error} \simeq \frac{1}{k_i x}.$$

This paper presents, in effect, a justification of methods used in previous experimental work on falling liquid jets. This work included measurements of the response of a jet to known excitation as well as measurements of the growth rate far from the exciter[7] and studies of the effects of external feedback loops on the stability of the jet.[17] In all of this work, the use of a complex wavenumber based on one of the averages (18) and (19) gave good agreement between experimental results and theories derived with the assumption of constant velocity.

ACKNOWLEDGMENTS

This work was supported in part by Air Force Grant 107-66, and in part by National Institutes of Health Grant PH-00-511. Part of this work was performed while the author was a North Atlantic Treaty Organization Postdoctoral Fellow.

[17] J. M. Crowley, Phys. Fluids 10, 1170 (1967).

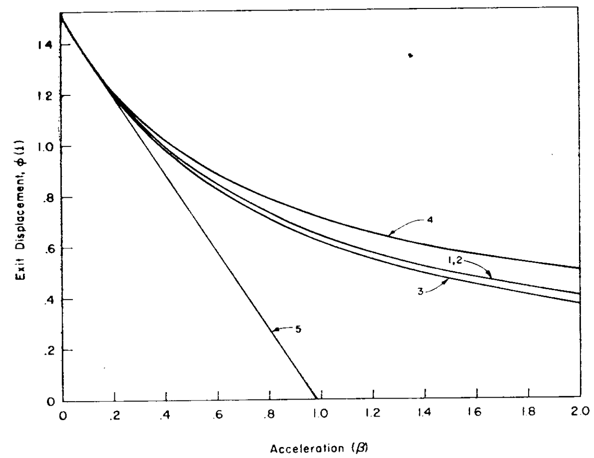

FIG. 6. Comparison of solutions for extreme growth ($N = 25$, $\omega = 0$, $R/d = 0.322$, $\alpha^2 = 0.01$).

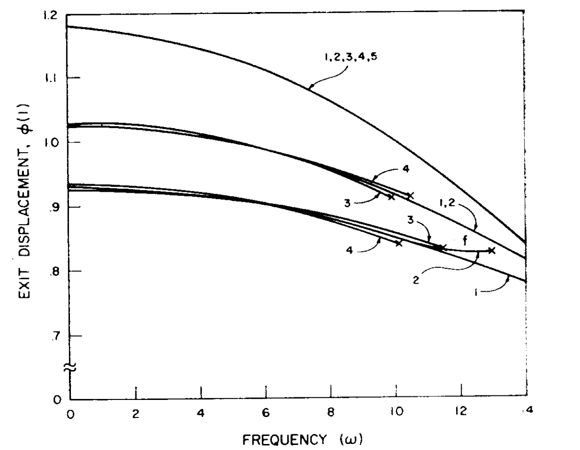

FIG. 7. Comparison of solutions as a function of frequency ($\omega_{\text{cutoff}} \simeq 10$, $\alpha^2 = 0.01$, $N = 1.0$).

three approximations give somewhat different results. The best is the integral mean [Eq. (17)] which differs less than 1% from the numerical solution for the most extreme case studied ($k_i \simeq 5$, $\beta = 2$). It appears that this approximation would be useful anywhere in this range.

The midpoint approximation [Eq. (18)] is somewhat inferior, differing from the exact solution by about 10% for $k_i \simeq 5$ and $\beta = 1$. For $k_i = 1$, however, the discrepancy is less than 1%, and this approximation, for practical purposes is equivalent to the numerical solution. The remaining approximation [Eq. (19)] which averages the initial and final growth rates, gives results inferior to the midpoint evaluation, and should not be used.

IX. AMPLITUDE VS FREQUENCY

The numerical solution is valid for all frequencies; the three approximations are not, since they assume an unstable disturbance growing at a rate k_i. At high frequencies the waves no longer grow, and the jet supports only stable propagating waves. Below this cutoff, however, the approximations give good agreement with the exact solution, as shown in Fig. 7 for $\beta = 1$. It appears from this figure, and other numerical results not shown, that the integral approximation [Eq. (17)] is as good over most growth frequencies ($0 \leq \omega \leq 0.9\omega_{\text{cutoff}}$) as at zero frequency. The accuracy of this approximation is not seriously affected by changing the values of R/d.

X. FAST GROWTH RATES

Far from the exciter, the exact nature of the excitation will be unimportant if the wave is growing, since the decaying part of the response will have died out. At great distances then, the magnitude of the disturbance is proportional to

$$\frac{e^{k_i x}}{k_i}\left(\frac{(1+\beta x)^{\frac{1}{2}}-1}{\frac{1}{2}\beta}\right).$$

If the amplitude is now measured at two points downstream, their ratio is

$$\frac{\hat{\delta}(x+\Delta x)}{\hat{\delta}(x)} = \frac{[1+\beta(x+\Delta x)]^{\frac{1}{2}}-1}{(1+\beta x)^{\frac{1}{2}}-1}\left(\frac{x}{x+\Delta x}\right)\exp\,[k_i(\Delta x)]$$

or

$$\ln\left[\frac{\hat{\delta}(x+\Delta x)}{\hat{\delta}(x)}\right] \simeq k_i(\Delta x) + \ln\left[\frac{[1+\beta(x+\Delta x)]^{\frac{1}{2}}-1}{(1+\beta x)^{\frac{1}{2}}-1}\left(\frac{x}{x+\Delta x}\right)\right]. \tag{22}$$

If the rate of change of the velocity is much smaller than the growth rate throughout the region, this equation can be expressed as

$$\frac{d\hat{\delta}}{\hat{\delta}} = k_i(x)\,dx, \tag{23}$$

where k_i is now evaluated at each point. This equation may be integrated to

$$\hat{\delta}(x+\Delta x) = \delta(x)\exp\,[(k_i)_{\text{eff}}\,\Delta x], \tag{24}$$

where

$$(k_i)_{\text{eff}} = \frac{1}{\Delta x}\int_x^{x+\Delta x} k_i(x)\,dx$$

and again an effective growth rate is given by the integral of $k_i(x)$. This expression is often used in must be modified to account for the changing velocity of the jet.

V. ANALYTICAL METHODS

The most straightforward approach is a numerical solution of the equation of motion (5) for given initial conditions. This method, however, yields little physical insight into the problem and makes a comparison with the unaccelerated jet very difficult. In addition, it is not always possible to express the behavior of the system in a single differential equation. In this paper the numerical solution will only be used occasionally to check the other approaches.

A more meaningful attack is a perturbation expansion in the space varying terms of the form

$$\hat{\delta} = \hat{\delta}_0 + \beta\hat{\delta}_1 + \cdots . \qquad (11)$$

Expanding the expressions $U(x)$ and $R(x)$ in terms of βx, substituting into the equation of motion (5), and collecting terms yield a series of equations which can be solved sequentially to give terms in the perturbation expansion. The first-order effect of the acceleration can easily be obtained by a comparison of the two leading terms in the expansion.

Finally, an approximate value of growth rate can be derived from the dispersion relation for the unaccelerated jet. This method is preferred for experimental work because it is simple and quick. The selection of effective growth constants will be guided by the results of the exact numerical solution and the perturbation expansion.

VI. PERTURBATION METHOD

In a growth measurement, only the magnitude of the disturbance if of interest. This magnitude is

$$|\hat{\delta}| = |\hat{\delta}_0 + \beta\hat{\delta}_1 + \cdots|$$

or, to first order in β

$$|\hat{\delta}| \simeq (\hat{\delta}_0^* \ \hat{\delta}_0)^{\frac{1}{2}}\left[1 + \frac{\beta}{2}\left(\frac{\hat{\delta}_1^*}{\hat{\delta}_0^*} + \frac{\hat{\delta}_1}{\hat{\delta}_0}\right)\right], \qquad (12a)$$

where $\hat{\delta}_0^*$ is the complex conjugate of $\hat{\delta}_0$. For the growing waves, the zero and first-order solutions take the form

$$\hat{\delta}_0 = R_0 e^{-ik_r x},$$

$$\hat{\delta}_1 = (R_1 + iI_1)e^{-ik_r x},$$

where R and I are real numbers. Substituting these expressions into Eq. (12a) yields

$$|\hat{\delta}| \simeq R_0 + \beta R_1 \qquad (12b)$$

to first order in β, therefore, only the real part of the solution is needed.

Substituting the perturbation expansion into the equation of motion, (4), and collecting terms yields, for the zero-order equation

$$(1 - \alpha_0^2)\frac{d^2\,\hat{\delta}_0}{dx^2} + 2i\omega\frac{d\,\hat{\delta}_0}{dx} - (N_0 + \omega^2)\,\hat{\delta}_0 = 0 \quad (13)$$

and for the first-order equation

$$(1 - \alpha_0^2)\frac{d^2\,\hat{\delta}_1}{dx^2} + 2i\omega\frac{d\,\hat{\delta}_1}{dx} - (N_0 + \omega^2)\,\hat{\delta}_1$$
$$= -(1 - 2\alpha_0\alpha_1)x\frac{d^2\,\hat{\delta}_0}{dx^2}$$
$$- \left(i\omega x + \frac{1 + \alpha_0^2}{2}\right)\frac{d\,\hat{\delta}_0}{dx} + N_1 x\,\hat{\delta}_0, \qquad (14)$$

where

$$N(\beta x) = N_0 + N_1\beta x + \cdots ,$$

$$\alpha(\beta x) = \alpha_0 + \alpha_1\beta x + \cdots .$$

The solution of the zero-order equation (11) which satisfies the boundary conditions (9) is

$$R_0 = \left(\frac{\sinh k_{i0}x}{k_{i0}x}\right)x. \qquad (15)$$

Substitution of this solution into the first-order equation (14), and solution gives for the real part of the first-order term

$$R_1 = -x^2\frac{(1 + \alpha_0^2/2)}{(1 - \alpha_0^2)}\frac{\sinh k_{i0}x}{4k_{i0}x}$$
$$+ \left[\frac{(1 - 2\alpha_0\alpha_1)(k_{r0}^2 - k_{i0}^2) - \omega k_{r0} + N_1}{4k_{i0}^2(1 - \alpha_0^2)}\right]$$
$$\cdot\left[x^2\cosh k_{i0}x - \frac{x\sinh k_{i0}x}{k_{i0}}\right], \qquad (16)$$

where k_{r0} and k_{i0} are the real and imaginary parts of the propagation constant calculated at the entrance ($x = 0$).

This result, which gives the amplitude of the disturbance in terms of its acceleration and physical properties, is only valid for small values of β, the acceleration parameter.

VII. APPROXIMATE DISPERSION RELATIONS

A third approach is to modify the expression which gives the displacement of the unaccelerated jet (15) to include the approximate effect of the acceleration by replacing the constant value of k_i by some average value of k_i over the region of interest. In this paper, three different averages will be used, short length of the jet (Fig. 2). The rate of change of transverse momentum for this section is (in rationalized mks units)

$$\rho\pi R^2(x) \frac{D^2 \delta}{Dt^2} \Delta x,$$

which equals the sum of the surface tension and electric forces. The surface tension force on the exposed ends is given by the product of the surface tension coefficient and the length of the line on which it acts, in this case the circumference of the jet. This force is

$$2\pi T \ R(x).$$

The total component of this force in the transverse direction is

$$2\pi T\left[\left(R \frac{\partial \delta}{\partial x}\right)\bigg|_{x+\Delta x} - \left(R \frac{\partial \delta}{\partial x}\right)\bigg|_{x}\right]$$

which reduces to

$$2\pi T\left(\frac{\partial R}{\partial x}\frac{\partial \delta}{\partial x} + R\frac{\partial^2 \delta}{\partial x^2}\right)$$

on a per unit length basis. The electric force per unit length on a grounded circular jet slightly displaced from the center of a cylindrical high-voltage electrode is[7]

$$\frac{2\pi\epsilon_0\Phi_0^2}{[d^2 - R^2(x)]\{\ln [d/R(x)]\}^2}\ \delta.$$

Here ϵ_0 is the permittivity of free space. Thus, the equation of motion is

$$\rho\pi R^2(x)\frac{D^2 \delta}{Dt^2} = 2\pi T\, R(x)\frac{\partial^2 \delta}{\partial x^2} + 2\pi T\frac{dR(x)}{dx}\frac{\partial \delta}{\partial x} + \frac{2\pi\epsilon_0\Phi_0^2}{[d^2 - R^2(x)]\{\ln [d/R(x)]\}^2}\ \delta. \qquad (1)$$

In most studies of growing waves, an assumed solution of the form $f(t)e^{ik_r x}$ is substituted into the equation of motion (1). The resulting ordinary differential equation is then solved for the time behavior in terms of the real wavenumber k_r. If the jet is not accelerated, this procedure indeed shows whether a disturbance of given wavelength is stable, but it is inappropriate here for two reasons: First, the solutions do not have a simple exponential form, due to the spatial variations of the velocity and radius; second, the wavelength of the disturbance cannot be specified by physical boundary conditions if the flow velocity is greater than the capillary wave phase velocity.[3] Instead the time behavior of the disturbance is determined by the upstream excitation, and we must solve for the spatial behavior. Therefore, we assume

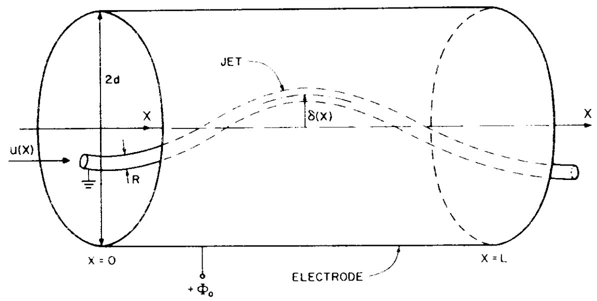

FIG. 1. The liquid jet, surrounded by a high-voltage electrode, accelerates in the x direction under gravity.

$$\delta = \hat{\delta}(x)e^{i\omega t}. \qquad (2)$$

The relation between these two approaches has been discussed by many authors.[9–12] The time derivative in the equation of motion (1) is the convective derivative, which may be written as

$$\frac{D\delta}{Dt} = \frac{\partial \delta}{\partial t} + V(x)\frac{\partial \delta}{\partial x}, \qquad (3)$$

correct to the first order in the perturbation. All lengths are now normalized to the length of the section under consideration, L, and all velocities to the jet velocity at the entrance of this section, $V(x = 0) = V_0$, so that

$$U(x) = V(x)/V_0.$$

All times are normalized to the ratio L/V_0. The equation of motion may now be written in normalized form as

$$[U^2(x) - \alpha^2(x)]\frac{d^2 \hat{\delta}}{dx^2} + \left\{2i\omega U(x) + U(x)\frac{dU(x)}{dx} - \alpha^2(x)\frac{d}{dx}[\ln R(x)]\right\}\frac{d\hat{\delta}}{dx} - [N(x) + \omega^2]\ \hat{\delta} = 0, \qquad (4)$$

where

$$N(x) = \frac{2\epsilon_0\Phi_0^2 L^2}{\rho U_0^2\, d^2 R^2(x)\{1 - [R(x)/d]^2\}\{\ln [d/R(x)]\}^2}, \qquad (5)$$

$$\alpha^2(x) = 2T[\rho\, R(x) U_0^2]^{-1}. \qquad (6)$$

From elementary mechanical considerations, the velocity of the falling jet is given by

$$U(x) = (1 + \beta x)^{\frac{1}{2}}, \qquad (7)$$

[9] R. Twiss, Proc. Phys. Soc. (London) **64B**, 654 (1951).
[10] P. Sturrock, Phys. Rev. **112**, 1488 (1958).
[11] M. Gaster, J. Fluid Mech. **14**, 222 (1962).
[12] R. J. Briggs, *Electron Stream Interactions with Plasmas* (The M.I.T. Press, Cambridge, Massachusetts, 1964), Chap. 2.